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Level dynamics for conservative and dissipative quantum systems

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Abstract. We establish level dynamics for finite matrices, employing a unified treatment of real symmetric, complex Hermitian, quaternion real, unitary, and arbitrary complex matrices. In all cases the level dynamics take the form of the classical Hamiltonian flow of some fictitious many-particle systems. Equilibrium statistical mechanics of the latter leads to the well known matrix ensembles of random-matrix theory. Ginibre's ensemble, in particular, is thus associated with level dynamics of arbitrary complex matrices.

1. Introduction

Level dynamics is the parametric motion of eigenvalues of finite matrices X such as

$$X(\lambda) = X_0 + \lambda Y_0. \tag{1.1}$$

The real parameter λ may be looked upon as the weight of a perturbation Y_0 of an 'initial' X_0. If Hermitian, such matrices arise as Hamiltonians of quantum systems and level dynamics is then just a fancy variant of perturbation theory for energy spectra.

Non-Hermitian matrices of the form (1.1) are encountered as generators of dissipative quantum dynamics, for instance in master equations $\mathrm{i}\mathrm{d}\rho/\mathrm{d}t = X\rho$ for density operators ρ. The complex eigenvalues of such generators have imaginary parts related to damping rates, while the real parts would reduce to energy differences in the conservative limit. Another application of non-Hermitian matrices is found in scattering theory: the complex poles of a unitary scattering matrix can be identified with the eigenvalues of a non-Hermitian X, with real and imaginary parts specifying locations and widths of resonances.

As observed already by von Neumann and Wigner in their seminal paper [1], different levels as a rule do not cross one another as a single control parameter such as λ is varied. That rule is ineffective when the matrix $X(\lambda)$ commutes with some other matrix Z for all λ. The eigenvalues of X then fall into different multiplets each of which is associated with a single eigenvalue of Z. While within one multiplet crossings are still generically avoided, levels from different multiplets are usually free to cross. When X generates the quantum-mechanical time evolution of some dynamical system, the commutativity $[X, Z] = 0$ means that Z is conserved. Such conservation laws are either related to symmetries or to integrability of the corresponding classical time evolution. Barring such exceptional cases, or confining ones attention to a single multiplet, one indeed encounters validity of

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the rule mentioned. To steer two levels to a degeneracy, one would have to control two or more parameters, and this is the situation we assume throughout the present paper.

It is well known that the level dynamics of Hermitian $N \times N$ matrices of the form (1.1) can be expressed as the classical Hamiltonian dynamics of a certain one-dimensional N-particle system [2–4]. That fictitious gas has λ as time and the real eigenvalues of X as particle coordinates. The Hamilton function $\mathcal{H}$ comprises a kinetic term as well as repulsive two-body interactions proportional to the inverse squared distance. Unusually, we consider the coupling strengths of the inverse-square potentials, which are themselves dynamical variables without analogues in real gases. Nevertheless, such many-body systems have found favour in the mathematical literature due to their integrability, prior to their independent discovery as an elegant formulation of level dynamics [5]. Alternatively, instead of considering variable coupling strengths (built from matrix elements of Y_0 in the eigenbasis of X), one can treat the eigenvectors of X as additional phase-space variables. The resulting dynamical equations, describing parametric motion of the eigenvalues, their rates of change, as well as eigenvectors and their Hermitian conjugates, have also Hamiltonian form, as observed by Nakamura and Lakshmanan [6]. Such systems were investigated earlier from the point of their complete integrability [7].

The equivalence of level dynamics with a fictitious many-particle system becomes more useful when the dependence of the $N \times N$ matrix X on the parameter is changed to [8]

$$X(\lambda) = X_0 \cos\lambda + Y_0 \sin\lambda. \tag{1.2}$$

In contrast to the matrix in (1.1), the modified one has levels not flying apart indefinitely, as the real parameter λ keeps growing. For the associated fictitious N-particle system, the modification amounts to the addition of a harmonic binding term for all N particles to the Hamilton function $\mathcal{H}$. The modified gas is thus confined, notwithstanding the repulsive interparticle interaction.

Upon application of equilibrium statistical mechanics to the many-particle system just mentioned, one arrives at a statistical theory of spectra of $N \times N$ matrices. The most naive attempt at a statistical treatment of the gas in consideration employs the canonical ensemble, i.e. the phase-space distribution $\exp(-\mathcal{H})$. For the eigenvalues of real symmetric matrices X, this turns out to yield a joint probability density identical to that well known from the Gaussian orthogonal ensemble (GOE) of random-matrix theory [9]; similarly, if the matrices in (1.2) are complex Hermitian or quaternion real, the canonical ensemble $\exp(-\mathcal{H})$ leads to the joint probability density of the Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE), respectively. To further corroborate the relation of random-matrix theory with equilibrium statistical mechanics of the many-body formulation of level dynamics, proper account of the integrability of the latter must be taken: as a consequence of the integrability, the phase-space point explores only a small sub-manifold of the energy surface. The appropriate equilibrium ensemble must therefore include further constants of the motion beyond just the energy $\mathcal{H}$ [4, 10, 11].

Our main objective in the present paper is to establish level dynamics for non-Hermitian $N \times N$ matrices of the form (1.1) and (1.2). Such matrices may represent generators of dissipative quantum dynamics [4] or, in another context, and after an appropriate modification, describe properties of scattering systems [12, 13]. We shall meet again with an equivalent classical Hamiltonian N-particle system, albeit one with the particles moving in a two-dimensional plane rather than along a line. The gas of particles must be two-dimensional simply because the eigenvalues of a non-Hermitian X are complex. The Hamilton function $\mathcal{H}$ encountered is real, positive and again includes, for the case (1.2), a harmonic confining potential. The associated canonical distribution, $\exp(-\mathcal{H})$, turns out to yield the joint

probability density of the N eigenvalues known from Ginibre's ensemble of random-matrix theory [14].

The conventional derivation of level dynamics of Hermitian matrices (1.1) or (1.2) employs eigenvectors $|\psi_n(\lambda)\rangle$ and eigenvalues $q_n(\lambda)$ of $X(\lambda)$: one differentiates

$$X(\lambda)|\psi_n(\lambda)\rangle = q_n(\lambda)|\psi_n(\lambda)\rangle$$

with respect to λ and multiplies it with the adjoint eigenvector $\langle\psi_m(\lambda)|$. The resulting set of differential equations is revealed as a classical Hamiltonian flow for the fictitious gas mentioned [4]. The conventional strategy *cannot* be applied to arbitrary complex matrices $X(\lambda)$. First, it can happen that such a matrix does not have N eigenvectors—in fact, the non-Hermitian generators of dissipative quantum dynamics are often of precisely this non-diagonalizable type. However, even more importantly, the obtained flow cannot be written as a Hamiltonian one with a *real* Hamilton function. If the Hamilton function is complex, the canonical ensemble cannot be sensibly constructed. The fact that a real Hamilton function does not exist for the obtained dynamical equations is connected to the impossibility of diagonalizing an arbitrary complex matrix by a unitary transformation—usually a general complex similarity transformation is needed to achieve this goal.

The best that can be achieved for non-diagonalizable matrices by a unitary transformation, U, is triangular form. The eigenvalues of X then appear as the diagonal elements $(U^{-1}XU)_{nn}$; all elements $(U^{-1}XU)_{mn}$ below the diagonal (i.e. the ones with $m > n$) vanish, but the $(U^{-1}XU)_{mn}$ with $m < n$ do not.

In order to establish the level dynamics we have to employ a more powerful method than the conventional one. First, the matrix motion (1.1) or (1.2) itself is revealed as classical Hamiltonian in character, within a phase space M, spanned by pairs of matrices X, Y. That Hamiltonian structure is then shown to be invariant under unitary transformations. Noether's theorem yields constants of the motion which in turn specify a sub-manifold of M to which the matrix dynamics (1.1) or (1.2) is actually restricted while still retaining Hamiltonian character. (The situation is analogous to the motion of a particle in a spherically symmetric potential; the conservation of angular momentum allows one to eliminate angular degrees of freedom and to establish a Hamiltonian sub-dynamics for the radial coordinate.) Exploiting the unitary invariance we choose the particular unitary transformation U, which brings about a triangular $U^{-1}XU$. The sub-manifold of M, to which the motion is confined, is parametrized so as to include the elements of $U^{-1}XU$ (and thus in particular the eigenvalues of X) among the coordinates. The change of coordinates in question is nonlinear, since the triangularization of X involves a unitary matrix U, depending on X. It remains to find the Hamilton function of the dynamics on the sub-manifold in question and the Poisson brackets of the new coordinates.

Conventional calculus is not a suitable tool for the task just outlined, due to the nonlinear change of coordinates, and the dimension $4N^2$ of M. Exterior differentiation and the theory of symplectic forms [15] finds a natural application here, and indeed renders all our calculations easy. We shall not assume the reader to have much familiarity with these techniques but rather develop the necessary machinery as we proceed.

More mathematical aspects of our investigation will be published elsewhere [16]. In particular, we shall employ there the so-called momentum mapping, to reduce Hamiltonian dynamics in a phase space M to a sub-manifold, due to symmetry [17, 18].

The present paper is organized as follows. In section 2 we exhibit the conventional construction of level dynamics for real symmetric matrices. As a build-up to our principal objective, we reconstruct this well known theory from our more general setting of Hamiltonian matrix dynamics. Exploiting the orthogonal invariance of the latter, we

adopt an orthogonal transformation which not only triangularizes but actually diagonalizes $X(\lambda)$. It is for this elementary and familiar case that we introduce the mathematical tools which become indispensable for arbitrary complex X in section 4. The intermediate section 3 briefly reports the extension to complex Hermitian, quaternion real, and even unitary matrices.

2. Level dynamics of real symmetric matrices

2.1. Elementary level dynamics

We propose consideration of dynamical equations for the eigenvalues of a real, symmetric $N \times N$ matrix X undergoing the parametric motion

$$\begin{aligned} &X(\lambda) = X_0 + \lambda Y_0 \\ &X_0 = X_0^{\mathrm{T}} = X_0^{\dagger} \qquad Y_0 = Y_0^{\mathrm{T}} = Y_0^{\dagger}. \end{aligned} \tag{2.1}$$

Such equations can be obtained from the eigenvalue equation, X

$$X|\psi_n(\lambda)\rangle = q_n(\lambda)|\psi_n(\lambda)\rangle \tag{2.2}$$

where $|\psi_n(\lambda)\rangle, n = 1, \ldots, N$, are the orthonormalized eigenvectors of X and q_n the corresponding eigenvalues (energies). By differentiating (2.2) with respect to λ and taking matrix elements between the eigenvectors $|\psi_n\rangle$, we obtain a closed system of differential equations for the quantities q_n, $p_n := \langle\psi_n|Y_0|\psi_n\rangle$, and $l_{mn} := \langle\psi_m|Y_0|\psi_n\rangle(q_n - q_m)$

$$\begin{aligned} \dot{q}_n &= \frac{\mathrm{d}q_n}{\mathrm{d}\lambda} = p_n \\ \dot{p}_n &= \frac{\mathrm{d}p_n}{\mathrm{d}\lambda} = -2\sum_{k\neq n} \frac{l_{nk}l_{kn}}{(q_n - q_k)^3} \\ \dot{l}_{mn} &= \frac{\mathrm{d}l_{mn}}{\mathrm{d}\lambda} = -\sum_{k\neq m,n} l_{mk}l_{kn}\left(\frac{1}{(q_m - q_k)^2} - \frac{1}{(q_n - q_k)^2}\right). \end{aligned} \tag{2.3}$$

The system (2.3) fully describes the fate of the eigenvalues when the parameter λ, a fictitious time, is changed. It takes a bit of ingenuity (see [2–4] for hints) to see that the system (2.3) is a Hamiltonian one. Indeed, when we define the following Poisson brackets for the phase-space variables q_n, p_n, and l_{mn} as

$$\begin{aligned} &\{p_m, q_n\} = \delta_{mn} \qquad \{p_m, p_n\} = 0 = \{q_m, q_n\} \\ &\{l_{mn}, p_i\} = 0 = \{l_{mn}, q_i\} \\ &\{l_{mn}, l_{ij}\} = \tfrac{1}{2}(\delta_{mj}l_{ni} + \delta_{ni}l_{mj} - \delta_{nj}l_{mi} - \delta_{mi}l_{nj}) \end{aligned} \tag{2.4}$$

the dynamical equations (2.3) turn out to be the Hamilton equations

$$\dot{q}_n = \{\mathcal{H}, q_n\} \qquad \dot{p}_n = \{\mathcal{H}, p_n\} \qquad \dot{l}_{mn} = \{\mathcal{H}, l_{mn}\} \tag{2.5}$$

with the Hamilton function

$$\mathcal{H}(q, p, l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_{n\neq m} \frac{l_{mn}^2}{(q_m - q_n)^2}. \tag{2.6}$$

One may consider the equations (2.3) or (2.5) as describing the time evolution of a one-dimensional gas of particles, with positions q_n and canonically conjugate momenta p_n, experiencing repulsive two-body interactions. A non-conventional feature of this fictitious gas is that the coupling strengths l_{mn} also undergo temporal changes according to the third

equation of (2.3). It must be noted that the gas in question must expand indefinitely, due to the interparticle repulsion and the lack of confining forces. Such expansion is in fact obvious from the parametric motion (2.1) of the matrix $X(\lambda) = X_0 + \lambda Y_0$: for Y generic, the eigenvalues of $X(\lambda)$ behave roughly like λ times the eigenvalues of Y_0, and thus fly apart indefinitely as $\lambda \to \infty$.

Not subject to such explosion are the levels of the real symmetric matrix

$$X(\lambda) = X_0 \cos\lambda + Y_0 \sin\lambda. \tag{2.7}$$

By analogy to the foregoing, this yields level dynamics with unchanged Poisson brackets, but the Hamilton function

$$\mathcal{H}(q, p, l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_n q_n^2 + \frac{1}{2}\sum_{n\neq m} \frac{l_{mn}^2}{(q_m - q_n)^2} \tag{2.8}$$

differs from (2.6) by an added harmonic confining potential. The change from the unbound parametric matrix motion (2.1) to the bounded one (2.7) can be understood as a non-canonical transformation of the fictitious-particle dynamics involving a λ-dependent re-scaling of the coordinates, as well as a nonlinear re-parametrization of the time λ [4, 8].

The fictitious gas with the binding Hamilton function (2.8) offers a convenient starting point for a statistical theory of the spectra of real symmetric matrices X. The energy surface $\mathcal{H} =$ constant is compact and yields a normalizable canonical phase-space density $\rho = \exp(-\mathcal{H})$. By integrating ρ over the variables p_n and l_{mn}, we obtain a density of the fictitious-particle coordinates, i.e. a joint density of the eigenvalues of X,

$$P(q_1, q_2, \ldots, q_n) \sim \exp\left(-\left(\frac{1}{2}\sum_n q_n^2\right)\right)\prod_{i<j}|q_i - q_j|. \tag{2.9}$$

This is in fact the distribution known from the Gaussian orthogonal ensemble of random-matrix theory [9], which appears here as a consequence of equilibrium statistical mechanics, for an associated many-body system.

2.2. Hamiltonian matrix dynamics

We shall now re-derive the Hamiltonian form of level dynamics for real symmetric matrices, using a different method which lends itself more easily to generalizations. The method employs a certain symmetry inherent in the equation (2.1), or rather in its differential form

$$\dot{X} = Y \qquad \dot{Y} = 0. \tag{2.10}$$

It is known from elementary analytical mechanics that symmetries of dynamical equations provide constants of the motion. For instance, rotational symmetry, as in a central-force problem, entails the conservation of angular momentum. Such constants of the motion enable us to reduce the number of degrees of freedom while preserving the Hamiltonian nature of the reduced system. In the case of a particle experiencing a central force, we can eliminate the angular degrees of freedom and reduce the Hamilton function to one depending only on the radial variable. In the case in hand, (2.10), the symmetry is an invariance under orthogonal transformations of the matrices X and Y.

The system of differential equations (2.10) looks like free motion in the space of pairs of real symmetric matrices (X, Y). It is therefore suggestive to assemble the matrix elements of the matrices X and Y into canonically conjugate pairs (X_{ij}, Y_{ij}) with $i \leqslant j$ of 'positions' X_{ij} and 'momenta' Y_{ij}, and to define the Poisson brackets

$$\{Y_{ii}, X_{kk}\} = \delta_{ik} \qquad \{Y_{ij}, X_{kl}\} = \tfrac{1}{2}\delta_{ik}\delta_{jl} \qquad \text{for } i < j, \quad k < l$$
$$\{Y_{ij}, Y_{kl}\} = 0 = \{X_{ij}, X_{kl}\}. \tag{2.11}$$

As is easily checked we thus reproduce the free motion (2.10) as the Hamilton equations

$$\dot{X}_{ij} = \{\mathcal{H}, X_{ij}\} \qquad \dot{Y}_{ij} = \{\mathcal{H}, Y_{ij}\} \qquad i, j = 1, \ldots, N \tag{2.12}$$

with the Hamilton function

$$\mathcal{H} = \frac{1}{2}\sum_i Y_{ii}^2 + \sum_{i<j} Y_{ij}^2 = \frac{1}{2}\mathrm{tr}Y^2. \tag{2.13}$$

To proceed, we need a slightly more abstract formulation of Hamiltonian mechanics. We furnish the phase space M (in the present example the differentiable manifold spanned by pairs of real symmetric matrices (X, Y)), with a differential 2-form ω which in our case reads

$$\omega = \sum_i \mathrm{d}Y_{ii} \wedge \mathrm{d}X_{ii} + 2\sum_{i<j} \mathrm{d}Y_{ij} \wedge \mathrm{d}X_{ij} = \mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X). \tag{2.14}$$

As in the second member of the foregoing equation, we usually omit the indices ij on X and Y and indicate summations as the trace operation; in employing this shorthand we must always keep in mind that only the X_{ij}, Y_{ij} with $i \leqslant j$ are our independent variables, and identify $X_{ij} = X_{ji}$.

In all cases to be considered, the symplectic 2-form ω will be the exterior derivative, $\omega = \mathrm{d}\alpha$, of some 1-form α, the so-called symplectic potential; in the present example this potential reads

$$\alpha = \mathrm{tr}(Y\,\mathrm{d}X). \tag{2.15}$$

The symplectic form ω can be used to define the Poisson brackets in a way which makes coordinate changes on M easy to implement. It is also convenient to use trace notation for a vector field $\mathcal{Z}$ with coefficients $\mathcal{Z}_X$ and $\mathcal{Z}_Y$

$$\mathcal{Z} = \mathrm{tr}\left(\mathcal{Z}_X \frac{\partial}{\partial X} + \mathcal{Z}_Y \frac{\partial}{\partial Y}\right). \tag{2.16}$$

The 1-form $\mathrm{d}\mathcal{F}$ associated with a smooth function $\mathcal{F}(X, Y)$ on M

$$\mathrm{d}\mathcal{F} = \mathrm{tr}\left(\frac{\partial \mathcal{F}}{\partial X}\,\mathrm{d}X + \frac{\partial \mathcal{F}}{\partial Y}\,\mathrm{d}Y\right) \tag{2.17}$$

entails the following change of $\mathcal{F}$ along $\mathcal{Z}$

$$d\mathcal{F}(\mathcal{Z}) = \mathrm{tr}\left(\mathcal{Z}_X \frac{\partial \mathcal{F}}{\partial X} + \mathcal{Z}_Y \frac{\partial \mathcal{F}}{\partial Y}\right). \tag{2.18}$$

If $\mathcal{G}$ is another smooth function on M, and $\mathrm{d}\mathcal{G}$ its differential, we construct a 2-form as $\mathrm{d}\mathcal{F} \wedge \mathrm{d}\mathcal{G}$. Acting with the latter on a pair of vector fields $\mathcal{X}$ and $\mathcal{Z}$, one obtains the function

$$(\mathrm{d}\mathcal{F} \wedge \mathrm{d}\mathcal{G})(\mathcal{X}, \mathcal{Z}) := \mathrm{d}\mathcal{F}(\mathcal{X})\,\mathrm{d}\mathcal{G}(\mathcal{Z}) - \mathrm{d}\mathcal{F}(\mathcal{Z})\,\mathrm{d}\mathcal{G}(\mathcal{X}) \tag{2.19}$$

which is obviously antisymmetric in the functions $\mathrm{d}\mathcal{F}(\mathcal{X})$, $\mathrm{d}\mathcal{G}(\mathcal{Z})$ defined in (2.18). If a pair of vector fields is to be acted upon by a linear combination of 2-forms such as $\sum_{\mu,\nu} \mathcal{A}_{\mu\nu}\mathrm{d}\mathcal{F}_\mu \wedge \mathrm{d}\mathcal{G}_\nu$ with smooth $\mathcal{A}_{\mu\nu}, \mathcal{F}_\mu$, and $\mathcal{G}_\nu$, then both sides of (2.19) must be correspondingly linearly combined.

The 'Hamiltonian vector field' $\mathcal{X}_\mathcal{F}$ of the smooth function $\mathcal{F}$ is defined by requiring that the action of the symplectic 2-form ω on the pair of vector fields $\mathcal{X}_\mathcal{F}$ and $\mathcal{Z}$ (with $\mathcal{Z}$ arbitrary) as (2.19) gives the negative change of $\mathcal{F}$ along $\mathcal{Z}$

$$\omega(\mathcal{X}_\mathcal{F}, \mathcal{Z}) = -\mathrm{d}\mathcal{F}(\mathcal{Z}). \tag{2.20}$$

Upon invoking the symplectic 2-form ω from (2.14) we find, from the foregoing definitions (2.18)–(2.20), the explicit expression

$$\mathcal{X}_{\mathcal{F}} = \mathrm{tr}\left(\frac{\partial \mathcal{F}}{\partial Y}\frac{\partial}{\partial X} - \frac{\partial \mathcal{F}}{\partial X}\frac{\partial}{\partial Y}\right). \tag{2.21}$$

We are now fully equipped to define the Poisson bracket $\{\mathcal{F}, \mathcal{G}\}$ of two functions by letting the pair of associated Hamiltonian vector fields be acted upon by the symplectic 2-form

$$\{\mathcal{F}, \mathcal{G}\} = \omega(\mathcal{X}_{\mathcal{F}}, \mathcal{X}_{\mathcal{G}}). \tag{2.22}$$

It is easy to check that this definition returns to the original Poisson brackets (2.11) when we employ the particular symplectic 2-form (2.14). We should note, however, that (2.20) and (2.22) make no explicit recourse to specific coordinates on M; and that this is why the slightly abstract formulation of Hamiltonian mechanics as a symplectic structure on M makes changes of coordinates easy to implement.

One can now observe that the symplectic form ω (2.14) (and hence also the Poisson brackets (2.11)) as well as the Hamilton function $\mathcal{H}$ (2.13) are invariant under orthogonal transformations

$$(X, Y) \longrightarrow (OXO^{-1}, OYO^{-1}) \qquad OO^{\mathrm{T}} = I \tag{2.23}$$

where the superscript T denotes matrix transposition. This symmetry under the group $O(N)$ comes with a constant of the motion

$$\mu(X, Y) = [Y, X]. \tag{2.24}$$

The conservation of the commutator is easily checked, $[Y, X] = [Y_0, X_0 + \lambda Y_0] = [Y_0, X_0]$. The relation of the conservation law to the symmetry (2.23) follows from Noether's theorem: let ξ be a real antisymmetric matrix and ϵ a small number; an orthogonal matrix O close to the unit matrix can then be written as $O = \exp(\epsilon\xi)$. For a Hamilton function to be invariant under such a transformation it must satisfy

$$\mathrm{tr}\left(\frac{\partial \mathcal{H}}{\partial X}[\xi, X] + \frac{\partial \mathcal{H}}{\partial Y}[\xi, Y]\right) = 0 \tag{2.25}$$

since $\delta X = \epsilon[\xi, X]$ and $\delta Y = \epsilon[\xi, Y]$ are the infinitesimal changes brought about by the transformation O in question. A generating function $\mathcal{G}$ producing such coordinate changes as $\delta X = \epsilon(\partial\mathcal{G}/\partial Y)$, $\delta Y = -\epsilon(\partial\mathcal{G}/\partial X)$ is easily found by integrating as

$$\mathcal{G} = \mathrm{tr}\,\xi[Y, X]. \tag{2.26}$$

The invariance of $\mathcal{H}$ under the infinitesimal transformation O thus implies $\{\mathcal{H}, \mathcal{G}\} = 0$ and this in turn means that $\mathcal{G}$ is invariant under the canonical transformations generated by the Hamilton function $\mathcal{H}$, i.e. that $\mathcal{G}$ is conserved. Now, since ξ is an arbitrary real antisymmetric matrix we may choose it such that only two elements are non-vanishing, $\xi_{ij} = \delta_{ik}\delta_{jl}$, and thus find $\mathcal{G}$ as $\mu_{kl} = [Y, X]_{kl}$; but this reasoning holds for all pairs of indices (kl) and therefore the whole matrix μ is conserved due to the $O(N)$-invariance of our canonical structure.

We now propose to exploit the $O(N)$-invariance to change coordinates on M, so as to obtain the eigenvalues of the matrix X among the new coordinates. To this end we choose the particular orthogonal transformation (2.23) which diagonalizes X

$$O^{-1}XO = \mathrm{diag}(q_1, q_2, \ldots, q_N) =: Q. \tag{2.27}$$

Obviously the matrix O is not uniquely determined by the matrix X. Nevertheless, we can always choose one matrix for each X such that the following construction of coordinates is correct.

We subject the matrices Y and μ to that same transformation and call the resulting matrices

$$P := O^{-1} Y O \qquad l := O^{-1} \mu O. \tag{2.28}$$

Needless to say, the so-adopted O depends on X such that we are now dealing with a nonlinear change of coordinates. Another consequence of the X-dependence of O is that the transform l of μ is not conserved.

In view of our goal, to rewrite our Hamiltonian dynamics (2.10) alias (2.12) in new coordinates we consider the differential of X. With the help of the 1-form

$$W := O^{-1}\,\mathrm{d}O = -W^{\mathrm{T}} \tag{2.29}$$

we get

$$\mathrm{d}X = \mathrm{d}(OQO^{-1}) = O(\mathrm{d}Q + [W, Q])O^{-1}. \tag{2.30}$$

We also need the differential $\mathrm{d}W$

$$\mathrm{d}W = \mathrm{d}(O^{-1}\,\mathrm{d}O) = -O^{-1}\,\mathrm{d}O\,O^{-1} \wedge \mathrm{d}O = -W \wedge W. \tag{2.31}$$

We can now express our symplectic potential α in the new coordinates

$$\alpha = \mathrm{tr}(Y\,\mathrm{d}X) = \mathrm{tr}(P(\mathrm{d}Q + [W, Q])) = \mathrm{tr}(P\,\mathrm{d}Q) - \mathrm{tr}(lW). \tag{2.32}$$

As its differential we have the symplectic 2-form

$$\begin{aligned}\omega = \mathrm{d}\alpha &= \mathrm{tr}(\mathrm{d}P \wedge \mathrm{d}Q) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W) \\ &= \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i + 2\sum_{i<j} \mathrm{d}l_{ij} \wedge W_{ij} \\ &\quad -2\sum_{i<j<k} (l_{ij}W_{jk} \wedge W_{ik} + l_{ik}W_{ij} \wedge W_{jk} + l_{jk}W_{ik} \wedge W_{ij})\end{aligned} \tag{2.33}$$

where p_i with $i = 1, \ldots, N$ are the diagonal elements of the matrix P, and l_{ij} are the elements of the antisymmetric matrix l.

As new coordinates on M we now choose N functions q_i, N functions p_i, $N(N-1)/2$ elements of the antisymmetric matrix l, and $N(N-1)/2$ independent coordinates parametrizing the orthogonal matrix O. A short calculation expresses the Hamilton function

$$\mathcal{H} = \tfrac{1}{2}\mathrm{tr}\,Y^2 = \tfrac{1}{2}\mathrm{tr}\,O^{-1}Y^2O \tag{2.34}$$

in terms of the new coordinates. Not surprisingly, we find exactly the Hamilton function of level dynamics (2.6). Moreover, upon using (2.20), (2.22), and (2.33) we derive the Poisson brackets in the new variables and recover those previously given in (2.4). We could proceed to calculate the Poisson brackets involving the remaining new phase-space variables, i.e. those characterizing the orthogonal matrix O. However, we refrain from doing this, because the Poisson brackets (2.4) for the p_i, q_i, l_{ij} close among themselves, and the Hamilton function does not involve these additional variables either. We can now enjoy an important benefit of the orthogonal symmetry of our matrix dynamics, i.e. the possibility of reducing the number of degrees of freedom, while preserving the Hamiltonian character.

The foregoing considerations carry over immediately to real symmetric matrices of the form (2.7), i.e. $X = X_0 \cos\lambda + Y_0 \sin\lambda$. Instead of the matrix dynamics (2.10) we now have

$$\dot{X} = Y \qquad \dot{Y} = -X. \tag{2.35}$$

This is Hamiltonian in character with the Poisson brackets (2.11) but with the Hamilton function (2.13) replaced by

$$\mathcal{H} = \sum_i (X_{ii}^2 + Y_{ii}^2) + \frac{1}{2}\sum_{i<j}(X_{ij}^2 + Y_{ij}^2) = \frac{1}{2}\mathrm{tr}(X^2 + Y^2). \tag{2.36}$$

Again there is symmetry under orthogonal transformations. Simply repeating the above analysis and by changing coordinates, we project the motion onto the smaller manifold spanned by the q, p, l and there recover Hamiltonian dynamics according to the fictitious gas Hamilton function (2.8), which includes a harmonic binding potential.

3. Level dynamics for Hermitian and unitary matrices

When extending the above arguments for Hamiltonian matrix dynamics to Hermitian $N \times N$ matrices or quaternion real $2N \times 2N$ matrices of the structure $X = X_0 + \lambda Y_0$ or $X = X_0 \cos\lambda + Y_0 \sin\lambda$, we encounter few changes. Noteworthy is the replacement of the orthogonal symmetry $O(N)$, by the unitary one $U(N)$ and the symplectic one $Sp(N)$, respectively. Consequently, the Poisson algebras of the l_{ij} are related to the corresponding Lie algebras $u(N)$ and $sp(N)$ rather than to the algebra $o(N)$ incurred in the previous section. Moreover, the matrix dynamics with binding potentials naturally lead to the joint distribution of eigenvalues of the Gaussian unitary and symplectic ensembles of random-matrix theory [9, 4].

Even the level dynamics of unitary Floquet matrices of periodically kicked quantum systems (see [4]) can be cast into the form of Hamiltonian matrix dynamics. Such Floquet matrices have the structure

$$F = \exp(-\mathrm{i}\lambda V)F_0 \tag{3.1}$$

with V Hermitian and F_0 unitary, both of dimension N, and λ once more a real control parameter. Of interest is the fate of the eigenphases upon varying λ.

In order to reveal the λ-dependence of Floquet matrices of the form (3.1) as a Hamiltonian flow we introduce a phase space M, spanned by pairs of $N \times N$ matrices (F, V) with F unitary and V Hermitian, or, equivalently, by pairs (X, Y) defined as

$$X = F \qquad Y = \mathrm{i}F^{-1}V. \tag{3.2}$$

For systems which enjoy a time reversal symmetry T, i.e. $TFT^{-1} = F^{\dagger}$ with an anti-unitary T, the Floquet matrix (3.1) is symmetric ($F = F^{\mathrm{T}}$), in the T-invariant basis and can be diagonalized by an orthogonal matrix O [4]. In this case we define the symplectic structure using the real form

$$\omega = \mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X) \tag{3.3}$$

and this together with the Hamilton function

$$\mathcal{H} = -\tfrac{1}{2}\mathrm{tr}(XY)^2 \tag{3.4}$$

leads to the dynamical equations

$$\dot{X} = -XYX \qquad \dot{Y} = YXY. \tag{3.5}$$

These Hamiltonian equations obey the conservation law, $XY = \text{constant} = X_0Y_0$, and hence the solution

$$X = \exp(-\lambda X_0 Y_0)X_0. \tag{3.6}$$

The latter is indeed, according to (3.2), the original unitary matrix $F(\lambda)$ in (3.1). Proceeding from here on as above, and exploiting the invariance of the Hamiltonian matrix dynamics, under orthogonal transformations we reduce the dynamics to the sub-manifold of M spanned by the fewer variables q_n, p_n and l_{mn}, which in the case of the orthogonal symmetry class obey the Poisson algebra (2.4) (when $U(N)$ or $Sp(N)$ is the reigning symmetry group we must modify the symplectic structure accordingly and then the Poisson brackets of the l_{mn} come out as related to the corresponding Lie algebras $u(N)$ or $sp(N)$). The new q_n are the eigenphases of the Floquet matrix F, while the canonically conjugate momenta p_n are the diagonal elements of the matrix

$$v = O^{\mathrm{T}} V O. \tag{3.7}$$

This is the original Hermitian matrix V rotated by the matrix O which diagonalizes F as

$$O^{\mathrm{T}} F O = \exp(-\mathrm{i}Q) \qquad Q = \mathrm{diag}(q_1, q_2, \ldots, q_N) \tag{3.8}$$

and the real antisymmetric matrix l is given by

$$l = \mathrm{i}(\exp(\mathrm{i}Q) v \exp(-\mathrm{i}Q) - v). \tag{3.9}$$

The Hamilton function (3.4) written in the coordinates q_n, p_n, and l_{mn} reads

$$\mathcal{H}(q, p, l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{8}\sum_{n,m;n\neq m} \frac{l_{mn}^2}{\sin^2((q_m - q_n)/2)}. \tag{3.10}$$

This leads to dynamical equations analogous to (2.3)

$$\begin{aligned}
\frac{\mathrm{d}q_n}{\mathrm{d}\lambda} &= p_n \\
\frac{\mathrm{d}p_n}{\mathrm{d}\lambda} &= -\frac{1}{4}\sum_{k\neq n} l_{nk} l_{kn} \frac{\cos((q_n - q_k)/2)}{\sin^3((q_n - q_k)/2)} \\
\frac{\mathrm{d}l_{mn}}{\mathrm{d}\lambda} &= -\frac{1}{4}\sum_{k\neq m,n} l_{mk} l_{kn} \left(\frac{1}{\sin^2((q_m - q_k)/2)} - \frac{1}{\sin^2((q_n - q_k)/2)} \right).
\end{aligned} \tag{3.11}$$

These Hamiltonian equations have previously been derived [4] by the more elementary method outlined in section 2.1. They obviously allow for an interpretation as the dynamics of a fictitious N-particle system. The equilibrium statistical mechanics of that many-body system has been shown to entail the statistics of the eigenphases q_n well known from Dyson's circular ensembles of random-matrix theory [4, 10, 11].

The special cases of constant l_{mn} in the Hamilton functions (2.6) and (3.10) give rise to the known Calogero–Moser and Sutherland–Moser systems [19–21]. It is interesting that the *quantum* versions of these Hamiltonians can also be associated with universal statistical spectral properties of quantum chaotic systems. It was shown by Simons *et al* [22] how quantum Calogero and Sutherland Hamiltonians appear in limiting ($N \to \infty$) versions of certain field-theoretical matrix models, which, on the other hand, lead to random matrix theory via the so-called nonlinear, supersymmetric σ-model [23, 24].

4. Level dynamics of complex matrices

4.1. Parametric motion of complex matrices as a real Hamiltonian flow

We now extend the techniques explained above to parametric motion in the space of complex matrices. Admitting complex $N \times N$ matrices, X and Y and a real parameter λ, we consider the motion

$$X(\lambda) = X_0 + \lambda Y \tag{4.1}$$

as a flow on the $2N^2$-dimensional complex phase space M, parametrized by pairs of matrices (X, Y). Alternatively and equivalently, the flow (4.1) can be written as a real one: instead of pairs of complex matrices (X, Y), we may choose quadruples of real ones, through $X = X^{(1)} + \mathrm{i}X^{(2)}$, $Y = Y^{(1)} + \mathrm{i}Y^{(2)}$ with $X^{(a)}$, $Y^{(a)}$, $a = 1, 2$; the phase space is then a $4N^2$-dimensional real manifold.

To reveal the Hamiltonian nature of the motion we introduce 'natural' Poisson brackets

$$\{Y_{ij}^{(a)}, X_{kl}^{(b)}\} = \delta_{ab}\delta_{ik}\delta_{jl} \qquad \{Y_{ij}^{(a)}, Y_{kl}^{(b)}\} = \{X_{ij}^{(a)}, X_{kl}^{(b)}\} = 0 \tag{4.2}$$

and a real positive Hamilton function

$$\mathcal{H} = \frac{1}{2}\mathrm{tr}YY^\dagger = \frac{1}{2}\sum_{i,j}^{N}(Y_{ij}^{(1)})^2 + \sum_{i,j}^{N}(Y_{ij}^{(2)})^2. \tag{4.3}$$

The Hamilton equations read

$$\dot{X}_{ij}^{(a)} = \{\mathcal{H}, X_{ij}^{(a)}\} = Y_{ij}^{(a)} \qquad \dot{Y}_{ij}^{(a)} = \{\mathcal{H}, Y_{ij}^{(a)}\} = 0. \tag{4.4}$$

Combining these to the complex equations $\dot{X} = Y, \dot{Y} = 0$ and integrating, we indeed recover the original flow (4.1). Needless to say, we could have stayed with complex matrices throughout and employed their Poisson brackets

$$\{Y_{ij}, X_{kl}^*\} = 2\delta_{ik}\delta_{jl} \qquad \{Y_{ij}, X_{kl}\} = \{Y_{ij}, Y_{kl}\} = \{Y_{ij}, Y_{kl}^*\} = \{X_{ij}, X_{kl}\} = \{X_{ij}, X_{kl}^*\} = 0. \tag{4.5}$$

From here on, we shall most often use the more compact formulation in terms of complex matrices.

In preparation of the envisaged change of coordinates, we equip the phase space M of pairs of complex matrices with the symplectic forms

$$\alpha = \mathrm{Re}\,\mathrm{tr}(Y\,\mathrm{d}X^\dagger) \qquad \omega = \mathrm{d}\alpha = \mathrm{Re}\,\mathrm{tr}(\mathrm{d}Y \wedge \mathrm{d}X^\dagger) \tag{4.6}$$

or, in terms of real matrices,

$$\alpha = \sum_{i,j} Y_{ij}^{(1)}\,\mathrm{d}X_{ij}^{(1)} + \sum_{i,j} Y_{ij}^{(2)}\,\mathrm{d}X_{ij}^{(2)} \qquad \omega = \sum_{i,j}\mathrm{d}Y_{ij}^{(1)} \wedge \mathrm{d}X_{ij}^{(1)} + \sum_{i,j}\mathrm{d}Y_{ij}^{(2)} \wedge \mathrm{d}X_{ij}^{(2)}. \tag{4.7}$$

We can now introduce Poisson brackets without reference to specific phase-space coordinates via the standard procedure. Let us recall from section 2 that one employs an arbitrary vector field $\mathcal{Z}$, on M, and associates a Hamiltonian vector field $\mathcal{X}_\mathcal{F}$, with an arbitrary smooth function $\mathcal{F}$, through $\omega(\mathcal{X}_\mathcal{F}, \mathcal{Z}) = -\mathrm{d}\mathcal{F}(\mathcal{Z})$. The Poisson bracket of two functions $\mathcal{F}$ and $\mathcal{G}$ is then defined as $\{\mathcal{F}, \mathcal{G}\} = \omega(\mathcal{X}_\mathcal{F}, \mathcal{X}_\mathcal{G})$. These, of course, imply the above brackets (4.2) or (4.5), by means of a straightforward calculation.

4.2. Invariance under the unitary group

The symplectic structure just introduced is symmetric under the following action of the unitary group $U(N)$

$$(X, Y) \longrightarrow (UXU^{-1}, UYU^{-1}) \qquad UU^\dagger = I. \tag{4.8}$$

This invariance is indeed easy to check for the symplectic form ω and the Hamilton function (4.3); the invariance of the Poisson brackets (4.2) or (4.5) follows from that of ω. As in the previous section, we find the constants of the motion

$$\mu(X, Y) := \tfrac{1}{2}([Y, X^\dagger] + [Y^\dagger, X]) \tag{4.9}$$

to be related to the symmetry in question through Noether's theorem. Moreover, we can exploit the invariance under $U(N)$ to introduce new coordinates on M so as to include the eigenvalues of the matrix X as a subset and thus exhibit level dynamics.

4.3. 'Rotating-frame' coordinates

It is not possible to diagonalize an arbitrary complex matrix X with the help of a unitary transformation. Fortunately, we need not diagonalize $X(\lambda)$ to explore its eigenvalues. It suffices to bring $X(\lambda)$ into upper-triangular form and that can indeed be achieved (see [25]) by a unitary transformation

$$Z := U^\dagger X U \qquad Z_{ij} = 0 \qquad \text{for } i > j \tag{4.10}$$

the diagonal elements Z_{ii} are the eigenvalues of X. The same transformation performed on the matrices Y and μ

$$P := U^\dagger Y U \qquad l := U^\dagger \mu(X, Y) U = \tfrac{1}{2}([P, Z^\dagger] + [P^\dagger, Z]) \tag{4.11}$$

yields (in general, full) complex matrices P and l.

Just as in the previous examples, $X = UZU^{-1}$ is not a unique parametrization of the matrix X. If U transforms X to the upper-triangular form (4.10) with the eigenvalues of X on the diagonal of the matrix Z, then UT with T, an arbitrary diagonal unitary matrix, does the same. We may thus impose N suitable constraints on U so as to be left with $N^2 - N$ independent real parameters in U. For instance, we could write $U = \exp[\mathrm{i}\sum_a x_a T_a]$ with the sum running over all non-diagonal $U(N)$ generators T_a. The real $N^2 - N$ coefficients x_a would be the coordinates uniquely specifying our triangularizing matrix U.

We now consider the differential of X,

$$\mathrm{d}X = \mathrm{d}(UZU^\dagger) = U(\mathrm{d}Z + [W, Z])U^\dagger. \tag{4.12}$$

Here, W is the 1-form

$$W := U^\dagger \,\mathrm{d}U = -W^\dagger. \tag{4.13}$$

From (4.13), and the unitarity of U, we obtain

$$\mathrm{d}W = \mathrm{d}(U^\dagger \mathrm{d}U) = -U^\dagger \,\mathrm{d}U U^\dagger \wedge \mathrm{d}U = -W \wedge W. \tag{4.14}$$

Hence, our symplectic potential α and the symplectic form ω can be written in the 'rotating coordinate frame' as

$$\begin{aligned} \alpha &= \mathrm{Re}\,\mathrm{tr}(Y\,\mathrm{d}X^\dagger) = \mathrm{Re}\,\mathrm{tr}(P(\mathrm{d}Z^\dagger + [Z^\dagger, W^\dagger])) = \mathrm{Re}\,\mathrm{tr}(P\,\mathrm{d}Z^\dagger) - \mathrm{tr}(lW) \\ \omega &= \mathrm{d}\alpha = \mathrm{Re}\,\mathrm{tr}(\mathrm{d}P \wedge \mathrm{d}Z^\dagger) - \mathrm{tr}(\mathrm{d}l \wedge W) + \mathrm{tr}(lW \wedge W). \end{aligned} \tag{4.15}$$

To replace the parametrization of the manifold M by pairs of complex matrices X, Y we now specify new coordinates: the triangular matrix Z, the upper-triangular part of P, i.e. the set of elements P_{ij} with $i \leqslant j$, the off-diagonal elements of the (anti-Hermitian) matrix l and the unitary matrix U. We thus have $N(N+1)$ real variables $\mathrm{Re}\,Z_{ij}$, $\mathrm{Im}\,Z_{ij}$ with $i \leqslant j$, $N(N+1)$ real variables $\mathrm{Re}\,P_{ij}$, $\mathrm{Im}\,P_{ij}$ with $i \leqslant j$, $N(N-1)$ real variables $\mathrm{Re}\,l_{ij}$, $\mathrm{Im}\,l_{ij}$, with $i < j$ (remember that $l_{ij} = -l^*_{ji}$) and $N(N-1)$ real variables parametrizing the triangularizing matrix U. As in the original frame we encounter $4N^2$ real variables. The transformation to rotating coordinates is generically non-singular. To fully implement it, we

must eliminate the variables P_{ij} with $i > j$ in favour of the variables l_{ij} with the help of the definition in (4.11). Writing out l_{mn} for $m < n$ and using $Z_{ij} = 0$ for $i > j$, we obtain

$$\begin{aligned} 2l_{mn} &= \sum_{k\geqslant n} P_{mk}Z^*_{nk} - \sum_{k\leqslant m} Z^*_{km}P_{kn} + \sum_{k\leqslant n} P^*_{km}Z_{kn} - \sum_{k\geqslant m} Z_{mk}P^*_{nk} \\ &= \sum_{k\geqslant n} Z^*_{nk}P_{mk} - \sum_{k\leqslant m} Z^*_{km}P_{kn} + \sum_{k\leqslant m} Z_{kn}P^*_{km} - \sum_{k\geqslant n} Z_{mk}P^*_{nk} \\ &\quad + \sum_{n\geqslant k>m} Z_{kn}P^*_{km} - \sum_{n>k\geqslant m} Z_{mk}P^*_{nk}. \end{aligned} \tag{4.16}$$

We may solve these equations for P^*_{mn} with $m > n$ and thus obtain the elements of the lower-triangular part of P as linear combinations of the l_{ij} with $i < j$ and the P_{ij} with $i \leqslant j$.

We shall need the Jacobian J (or rather the dependence of its absolute value on Z_{ij}) of the transformation from the variables ($\mathrm{Re}\,P_{mn}$, $\mathrm{Im}\,P_{mn}$, $m > n$) to the variables ($\mathrm{Re}\,l_{mn}$, $\mathrm{Im}\,l_{mn}$, $m \neq n$), or, what is equivalent up to a constant, of the transformation from $(P_{mn}, P^*_{mn}, m > n)$ to $(l_{mn}, l^*_{mn}, m > n)$. To construct J we deduce from (4.16) for $m < n$

$$\begin{aligned} 2l_{mn} &= \sum_{n\geqslant k>m} Z_{kn}P^*_{km} - \sum_{n>k\geqslant m} Z_{mk}P^*_{nk} + \cdots \\ &= (Z_{nn} - Z_{mm})P^*_{nm} + \sum_{n>k>m} (Z_{kn}P^*_{km} - Z_{mk}P^*_{nk}) + \cdots \end{aligned} \tag{4.17}$$

where the dots refer to terms independent of P_{ij}, P^*_{ij} with $i > j$ and thus incapable of contributing to the Jacobian. Obviously, then, the l_{mn} with $m < n$ do not depend on the P_{ij} with $i > j$ but only on their conjugates P^*_{ij}; hence the absolute value of the Jacobian J equals the squared modulus of the Jacobian J_1 of the transformation (4.17). Now, for $n = m + 1$, we have only one term on the right-hand side of (4.17) which contributes the factor $(Z_{m+1,m+1} - Z_{mm})$ to J_1. We can expand J_1 with respect to the rows containing only one element of this form. In the reduced determinant there are once more rows containing only one element corresponding to $n = m + 2$; thus by continuing this procedure we arrive at $J_1 = \prod_{i<j}(Z_{jj} - Z_{ii})$ and consequently

$$J = \prod_{i<j} |Z_{jj} - Z_{ii}|^2. \tag{4.18}$$

It is worth noting that the Jacobian J depends only on the diagonal elements Z_{ii}, i.e. the eigenvalues of X (and their complex conjugates).

With the help of the symplectic form (4.15) we can calculate the Poisson brackets for the new coordinates. To this end we first introduce the latter in ω as

$$\omega = \frac{1}{2}\sum_{i\leqslant j} \left(\mathrm{d}P_{ij} \wedge \mathrm{d}Z^*_{ij} + \mathrm{d}P^*_{ij} \wedge \mathrm{d}Z_{ij}\right) + \mathrm{tr}(\mathrm{d}l \wedge W) - \mathrm{tr}(lW \wedge W). \tag{4.19}$$

Mere inspection reveals the Poisson bracket between Z and P to be canonical,

$$\{P_{ij}, Z^*_{kl}\} = 2\delta_{ik}\delta_{jl} \qquad \{P_{ij}, Z_{kl}\} = \{Z_{ij}, Z^*_{kl}\} = \{P_{ij}, P^*_{kl}\} = 0 \tag{4.20}$$

and the Poisson brackets of P_{ij} and Z_{ij} with l_{kl} and U_{kl} to vanish. Moreover, the Poisson brackets for the l_{ij} do not depend on the U_{kl}, in analogy to the situation encountered in section 2.2. Indeed, let $\mathcal{F}$, $\mathcal{G}$ be two functions depending on the components of l only and $\mathcal{X}_\mathcal{F}$, $\mathcal{X}_\mathcal{G}$ the corresponding Hamiltonian vector fields calculated according to (2.20). A straightforward calculation (see the appendix) shows that

$$\mathcal{F}, \mathcal{G}\} = \omega(\mathcal{X}_\mathcal{F}, \mathcal{X}_\mathcal{G}) = -\mathrm{tr}\left(l\left[\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}, \left(\frac{\partial\mathcal{G}}{\partial l}\right)^{\mathrm{T}}\right]\right) \tag{4.21}$$

where $\partial\mathcal{F}/\partial l$ is the matrix with the elements $\partial\mathcal{F}/\partial l_{ij}$ etc. In particular, for the l_{ij} themselves we obtain the brackets related to the Lie algebra, $u(N)$

$$\{l_{pq}, l_{mn}\} = l_{pn}\delta_{mq} - l_{mq}\delta_{pn}. \tag{4.22}$$

In order to express the Hamilton function (4.3) in the new coordinates we start from

$$\mathcal{H}(Z, P', l') = \frac{1}{2}\mathrm{tr}(PP^\dagger) = \frac{1}{2}\left\{\sum_{i\leqslant j}|P_{ij}|^2 + \sum_{i>j}|P_{ij}|^2\right\} \tag{4.23}$$

and here think of the P_{ij} with $i > j$ as expressed in terms of the Z_{ij}, the off-diagonal elements of l and the upper-triangular part of P (which are denoted by l' and P', respectively). Together with the Poisson brackets (4.20) and (4.22), this Hamilton function yields the equations of motion for the variables Z, P' and l'. Since $\mathcal{H}$ does not depend on U, it is obvious that the latter variables obey a closed system of dynamical equations. Similar to our procedure in section 2, we have thus achieved the desired projection of the original matrix dynamics onto a Hamiltonian system of smaller dimension, such that the levels of the complex matrix $X(\lambda)$ span a part of the reduced manifold.

The reduced Hamiltonian dynamics arrived at, generalizes the previously known level dynamics to that of complex matrices. It may be looked upon as the dynamics of a 'gas' of fictitious particles moving in two spatial dimensions and we shall, in fact, dwell on that analogy below. Like the one-dimensional fictitious gases associated with Hermitian and unitary matrices in sections 1 and 2 the many-body system now encountered is an integrable one. The integrability is a trivial consequence of that of the original matrix flow (4.1). Needless to say, if we restrict the matrices X, Y in (4.1) to real symmetric ones, the new many-body system reduces to the corresponding one of section 2: the upper-triangular matrix Z becomes diagonal, the matrix P real symmetric, and the matrix l real antisymmetric with (4.17) specialized to the form $l_{mn} = P_{mn}(Z_{nn} - Z_{mm})$, familiar from section 2, apart from its notation.

4.4. Re-scaled levels and Ginibre's ensemble

The eigenvalues Z_{ii} of the complex matrix $X = X_0 + \lambda Y$ tend to move apart indefinitely in the complex plane as the real parameter λ keeps growing. We therefore shift our attention to a matrix dynamics with bounded motion of the eigenvalues and return to (2.7) and (2.35)

$$\dot{X} = Y \qquad \dot{Y} = -X \qquad \Leftrightarrow \qquad X = X_0\cos\lambda + Y_0\sin\lambda \tag{4.24}$$

but now with complex $N \times N$ matrices. That matrix flow is generated by the Hamilton function

$$\mathcal{H} = \tfrac{1}{2}\mathrm{tr}(XX^\dagger) + \tfrac{1}{2}\mathrm{tr}(YY^\dagger) \tag{4.25}$$

or, written in real variables,

$$\mathcal{H} = \frac{1}{2}\sum_{ij}^{N}\left(X_{ij}^{(1)}\right)^2 + \frac{1}{2}\sum_{ij}^{N}\left(X_{ij}^{(2)}\right)^2 + \frac{1}{2}\sum_{ij}^{N}\left(Y_{ij}^{(1)}\right)^2 + \frac{1}{2}\sum_{ij}^{N}\left(Y_{ij}^{(2)}\right)^2. \tag{4.26}$$

This differs from the original Hamilton function (4.3)) by the inclusion of a harmonic binding potential. That potential preserves the unitary symmetry of the system and hence all previous considerations remain valid. In the new coordinates the Hamilton function (4.23) acquires a binding term as well and takes the form

$$\mathcal{H}(Z, P', l') = \tfrac{1}{2}\mathrm{tr}(ZZ^\dagger) + \tfrac{1}{2}\mathrm{tr}(PP^\dagger). \tag{4.27}$$

Now that the motion of the eigenvalues in the complex plane is bounded in character, we have reached, as in section 1 for real symmetric matrices, a good basis for a statistical theory of spectra for complex matrices. Thinking of the level dynamics as the motion of a many-body system and imagining that system in thermal equilibrium, we employ the canonical phase-space density

$$\rho(Z, P', l') \sim \exp(-\mathcal{H}(Z, P', l') = \exp\left(-\frac{1}{2}\sum_{n}^{N}|Z_{nn}|^2\right)\exp\left(-\frac{1}{2}\sum_{m,n;m<n}^{N}|Z_{mn}|^2\right)\exp\left(-\frac{1}{2}\mathrm{tr}(PP^\dagger)\right). \tag{4.28}$$

Since the manifold $\mathcal{H} =$ constant is compact, this canonical density is normalizable. Of course, we must, as previously, imagine the elements from the lower-triangular part of P expressed by appropriate elements of l, Z and the upper-triangular part of P. The reduced distribution of the eigenvalues is obtained by integrating $\rho(Z, P', l')$ over the variables l', P', and $Z_{ij}, i < j$, i.e. all variables except the eigenvalues Z_{ii}

$$P(\{Z_{ii}\}) \sim \int \rho(Z, P', l') \prod_{i<j}^{N} \mathrm{d}^2 Z_{ij}\, \mathrm{d}^2 P_{ij}\, \mathrm{d}^2 l_{ij} \prod_{i}^{N} \mathrm{d}^2 P_{ii} \tag{4.29}$$

where $\mathrm{d}^2 x := \mathrm{dRe}\, x\, \mathrm{dIm}\, x$. The integration over $\mathrm{d}Z_{ij}, i < j$, involves only Gaussian factors stemming from $\mathrm{tr}XX^\dagger$ in the exponential. The integrals over the remaining variables are also reduced to Gaussian ones over $\exp(\mathrm{tr}\, PP^\dagger)$, if we change back from the variables l', P' to P. Taking into account the Jacobian (4.18) we obtain

$$P(\{Z_{ii}\}) \sim \exp\left(-\sum_{i}^{N}|Z_{ii}|^2\right)\prod_{i<j}|Z_{ii} - Z_{jj}|^2 \tag{4.30}$$

which is well known as the joint density of eigenvalues of Ginibre's ensemble [14] of random complex matrices.

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Appendix. Poisson brackets

We derive here the Poisson brackets (4.21), related to the symmetry group $U(N)$. A similar calculation would yield the corresponding brackets in (2.4) for the group $O(N)$. In fact, the construction is based on a general principle which follows from the equivariance of the moment map and the existence of the canonical Poisson structure on the algebra in question. The reader will appreciate the power of exterior differential forms as a bookkeeping device in changes of coordinates on some manifold. To do this kind of calculation using conventional calculus would be possible, in principle, but considerably more cumbersome.

The first step is to construct the Hamiltonian vector field $\mathcal{X}_\mathcal{F}$ associated with a smooth function $\mathcal{F}(l)$, which depends only on the l_{ij} and thus has the differential

$\mathrm{d}\mathcal{F} = \mathrm{tr}(\partial\mathcal{F}/\partial l)\,\mathrm{d}l^{\mathrm{T}}$. The change of $\mathcal{F}$, along an arbitrary vector field $\mathcal{Z}$, therefore involves only the components $\mathcal{Z}^{(l)}$ of $\mathcal{Z}$ along the l and reads

$$\mathrm{d}\mathcal{F}(\mathcal{Z}) = \mathrm{tr}\,\mathcal{Z}^{(l)}\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}. \tag{A.1}$$

We now recall the definition (2.20) of a Hamiltonian vector field, $\omega(\mathcal{X}_{\mathcal{F}}, \mathcal{Z}) = -\mathrm{d}\mathcal{F}(\mathcal{Z})$, and invoke the symplectic form (4.19). Since (i) $\omega(\mathcal{X}_{\mathcal{F}}, \mathcal{Z})$ is bilinear in the components of $\mathcal{X}_{\mathcal{F}}$ and $\mathcal{Z}$, (ii) $\mathrm{d}\mathcal{F}(\mathcal{Z})$ involves only the l-components $\mathcal{Z}^{(l)}$ of $\mathcal{Z}$, and (iii) ω has no cross terms between the $\mathrm{d}Z_{ij}$ and the $\mathrm{d}l_{ij}$ nor between the $\mathrm{d}P_{ij}$ and the $\mathrm{d}l_{ij}$, the Hamiltonian vector field $\mathcal{X}_{\mathcal{F}}$ in search can neither have Z- nor P-components; possible only are l- and U-components. We thus get

$$\begin{aligned}\omega(\mathcal{X}_{\mathcal{F}}, \mathcal{Z}) &= -\mathrm{tr}\,(\mathrm{d}l \wedge \mathrm{d}W - lW \wedge W)(\mathcal{X}_{\mathcal{F}}, \mathcal{Z}) \\ &= -\mathrm{tr}\,(\mathrm{d}l \wedge \mathrm{d}W - lW \wedge W)\Bigg(\mathrm{tr}\,\mathcal{X}_{\mathcal{F}}{}^{(l)}\left(\frac{\partial}{\partial l}\right)^{\mathrm{T}} \\ &\quad + \mathrm{tr}\,\mathcal{X}_{\mathcal{F}}{}^{(U)}\left(\frac{\partial}{\partial U}\right)^{\mathrm{T}},\ \mathrm{tr}\,\mathcal{Z}^{(l)}\left(\frac{\partial}{\partial l}\right)^{\mathrm{T}} + \mathrm{tr}\,\mathcal{Z}^{(U)}\left(\frac{\partial}{\partial U}\right)^{\mathrm{T}}\Bigg).\end{aligned} \tag{A.2}$$

Upon invoking the definition (2.19) of the action of a 2-form on a pair of vector fields we proceed to

$$\begin{aligned}\omega(\mathcal{X}_{\mathcal{F}}, \mathcal{Z}) &= -\mathrm{tr}\,(\mathcal{X}_{\mathcal{F}}{}^{(l)}U^{\dagger} - lU^{\dagger}\mathcal{X}_{\mathcal{F}}{}^{(U)}U^{\dagger} + U^{\dagger}\mathcal{X}_{\mathcal{F}}{}^{(U)}lU^{\dagger})\mathcal{Z}^{(U)} + \mathrm{tr}\,U^{\dagger}\mathcal{X}_{\mathcal{F}}{}^{(U)}\mathcal{Z}^{(l)} \\ &= -\mathrm{d}\mathcal{F}(\mathcal{Z}) = -\mathrm{tr}\,\mathcal{Z}^{(l)}\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}.\end{aligned} \tag{A.3}$$

We compare here the coefficients of $\mathcal{Z}^{(l)}$ and $\mathcal{Z}^{(U)}$ and obtain the desired vector field,

$$\mathcal{X}_{\mathcal{F}} = -\mathrm{tr}\Bigg(\left[l, \left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}\right]\left(\frac{\partial}{\partial l}\right)^{\mathrm{T}} + U\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}\left(\frac{\partial}{\partial U}\right)^{\mathrm{T}}\Bigg). \tag{A.4}$$

The Poisson bracket between two functions $\mathcal{F}(l)$ and $\mathcal{G}(l)$ is then accessible through the general definition (2.22), i.e. by letting the symplectic form ω act on the pair of associated Hamiltonian vector fields $\mathcal{X}_{\mathcal{F}}$ and $\mathcal{X}_{\mathcal{G}}$. We simply repeat the foregoing calculation with $\mathcal{Z}$ replaced by $\mathcal{X}_{\mathcal{G}}$ and recover the result announced in (4.21)

$$\{\mathcal{F}, \mathcal{G}\} = \omega(\mathcal{X}_{\mathcal{F}}, \mathcal{X}_{\mathcal{G}}) = -\mathrm{tr}\Bigg(l\left[\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}}, \left(\frac{\partial\mathcal{G}}{\partial l}\right)^{\mathrm{T}}\right]\Bigg). \tag{A.5}$$

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